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1992 J. Phys. A: Math. Gen. 25 L1343

(http://iopscience.iop.org/0305-4470/25/24/001)

J. Phys. A: Math. Gen. 25 (1992) L1343–L1349. Printed in the UK

LETTER TO THE EDITOR

The full set of c_n-invariant factorized S-matrices

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Received 8 July 1992

Abstract. We use the method of the tensor product graph to construct rational (Yangian invariant) solutions of the Yang–Baxter equation in fundamental representations of c_n and thence the full set of c_n-invariant factorized S-matrices.

Integrable quantum field theories in $1+1$ dimensions are expected to have exact S-matrices in which particle number and the set of asymptotic momenta are conserved, and in which multi-particle interactions factorize into products of two-particle interactions. The condition that this factorization be consistent is then the Yang–Baxter equation (YBE), so that in theories with a global Lie group invariance, such as the principal chiral model, the S-matrices are constructed from group-invariant solutions of the YBE. The spectrum of the theory then consists of multiplets within which the particles have equal mass and which form representations of the group G.

One method for constructing these S-matrices is the bootstrap procedure (known as the 'fusion procedure' for solutions of the YBE) in which, at appropirate poles, intermediate states of the S-matrix are identified as particle states whose S-matrices can then be constructed. An alternative method is to construct explicitly the action of the underlying charge algebra on particle multiplets, and then use conservation of these charges to deduce the S-matrix. It has become clear [1] that this algebra is precisely Drinfeld's Yangian [2] $Y(\mathcal{A})$, where $\mathcal{A}$ is the Lie algebra of the group G: if we write the action of the charges on states consisting of two asymptotically free particles as Δ, there is a local charge satisfying

$$[Q_0^a, Q_0^b] = \mathrm{i}\hbar f^{abc} Q_0^c \qquad \text{and} \qquad \Delta(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a \tag{1}$$

where f^{abc} are the structure constants of $\mathcal{A}$, and a series of non-local charges, the first of which satisfies

$$[Q_0^a, Q_1^b] = \mathrm{i}\hbar f^{abc} Q_1^c \qquad \text{and} \qquad \Delta(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a + \tfrac{1}{2} f^{abc} Q_0^c \otimes Q_0^b. \tag{2}$$

The requirement that Δ be a homomorphism (i.e. that asymptotic states carry representations of the charge algebra) fixes‡

$$f^{d[ab}[Q_1^{c]}, Q_1^d] = (i\hbar/12) f^{api} f^{bqj} f^{crk} f^{ijk} Q_0^{(p} Q_0^q Q_0^{r)} \tag{3}$$

where [ ] and ( ) denote (anti-)symmetrization on the enclosed indices.

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‡ For $\mathcal{A} \neq sl(2)$. For the general condition see Drinfeld [2].

This charge algebra provides a dynamical symmetry since it does not commute with the Poincaré group: if the rapidity (defined by $p=(m\cosh\theta, m\sinh\theta)$) of a state is given a Lorentz boost L_θ of rapidity θ, it is found that

$$Q_0^a \mapsto Q_0^a \qquad \text{and} \qquad Q_1^a \mapsto Q_1^a - \frac{\hbar C_{\text{Adj}}}{4\pi}\theta Q_0^a \tag{4}$$

where $C_{\text{Adj}}\delta_{ad}=f^{abc}f^{cbd}$ gives the value of the quadratic Casimir operator $C_2=Q_0^aQ_0^a$ in the adjoint representation. The S-matrix is thus constrained by conservation of Q_0^a and Q_1^a, so that the interaction $S(\theta_1-\theta_2)$ of two particles of rapidities θ_1 and θ_2 satisfies

$$[S(\theta_1-\theta_2), Q_0^a\otimes 1+1\otimes Q_0^a]=0 \tag{5}$$

$$S(\theta_1-\theta_2)(L_{\theta_1}\otimes L_{\theta_2}\Delta(Q_1^a))=(L_{\theta_2}\otimes L_{\theta_1}\Delta(Q_1^a))S(\theta_1-\theta_2) \tag{6}$$

which together imply the Yang–Baxter equation

$$\begin{aligned}(S(\theta_2-\theta_3)\otimes 1)(1\otimes S(\theta_1-\theta_3))(S(\theta_1-\theta_2)\otimes 1)\\ =(1\otimes S(\theta_1-\theta_2))(S(\theta_1-\theta_3)\otimes 1)(1\otimes S(\theta_2-\theta_3)).\end{aligned}$$

This acts on a state consisting of three asymptotically free particles of rapidities θ_1, θ_2 and θ_3, with each factor giving an interaction between two of the three particles. The S-matrix is related to the usual Yangian R-matrix by $S=\mathbf{P}R$, where $\mathbf{P}$ transposes states in a tensor product.

The particle multiplets of the theory are then irreducible representations of $Y(\mathscr{A})$ which may or may not be irreducible as representations of the Lie subalgebra $\mathscr{A}$, (1). In particular, an irreducible representation V of $\mathscr{A}$ may be extended to a representation v of $Y(\mathscr{A})$, with

$$\rho_v(Q_0)=\rho_V(Q_0) \qquad \text{and} \qquad \rho_v(Q_1)=0 \tag{7}$$

provided the action of the right-hand side of (3) vanishes on V, and Drinfeld classified those V for which this is true [2]. In particular, it is true of all the fundamental representations of a_n and c_n, but of only a few of the fundamental representations of other algebras. Since we expect the particle multplets to be fundamental representations of $Y(\mathscr{A})$ (i.e. those representations of $Y(\mathscr{A})$ containing a fundamental representation of $\mathscr{A}$ as a component), we therefore have an explicit action of the charge algebra on all the particle multiplets of the a_n and c_n theories: the charges have action (7) on particles of zero rapidity, boosted by (4) on a particle of rapidity θ.

A method then exists [3–5] for solving (5), (6). For full details we refer the reader to the original papers, but in brief one first uses the group invariance (5) of $S(\theta)$ and Schur's lemma to give, for irreducible $\mathscr{A}$-representations X and Y where $X\otimes Y$ contains no multiplicities,

$$S_{XY}(\theta)=\mathbf{P}\sum_{W\subset X\otimes Y}\tau_W(\theta)P_W$$

where the P_W project onto irreducible components of $X\otimes Y$. We next use

$$\tfrac{1}{2}[C_2, 1\otimes Q_0^a]=f^{abc}Q_0^c\otimes Q_0^b$$

and projecton the left and right of (6) with P_R and P_S to obtain

$$\frac{\tau_S(u)}{\tau_R(u)}=\frac{\theta+\Delta_{RS}}{\theta-\Delta_{RS}} \tag{8}$$

where

$$\Delta_{RS}=\frac{C_2(R)-C_2(S)}{C_{\mathrm{Adj}}}\,\mathrm{i}\pi$$

for those R, S which have opposite parity in $X\otimes Y$ and for which $R\subset \mathrm{adjoint}\otimes S$. This system of equations is made most transparent by letting the components of $X\otimes Y$ be the nodes of a bipartite graph, joined by directed edges $R\to S$ labelled with the differences of the Casimir operators when there is a corresponding equation (8). That the system is consistent is then equivalent to the requirement that the set of labels on the edges be the same for all paths between two nodes of the graph. As expected from Drinfeld's results, this is found to be the case for all the fundamental representations of a_n and c_n. The solutions for the a_n case were obtained by Kulish *et al* [3] using both the fusion procedure and a method similar to this, but for c_n only a few solutions have been found: with the fundamental representations of c_n labelled by

1 2 3 $n-2$ $n-1$ n

the bootstrap procedure has been used [4] to construct $S_{1m}(\theta)$ starting from the $S_{11}(\theta)$ previously calculated [6]. However, we can now construct the full set $S_{lm}(\theta)$ using the tensor product graph (TPG), which for $l\otimes m$, $l\geqslant m$ is

$$\begin{array}{ccccccccc}
\lambda_l+\lambda_m & \to & \lambda_{l+1}+\lambda_{m-1} & \cdots\to & \lambda_n+\lambda_{l+m-n} & \cdots\to & \lambda_{l+m-1}+\lambda_1 & \to & \lambda_{l+m}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\lambda_{l-1}+\lambda_{m-1} & \to & \lambda_l+\lambda_{m-2} & \cdots\to & \lambda_{n-1}+\lambda_{l+m-n-1} & \cdots\to & \lambda_{l+m-2} & & \\
\vdots & & \vdots & & \vdots & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\lambda_{n-m}+\lambda_{n-l} & \to & \lambda_{n-m+1}+\lambda_{n-l-1} & \cdots\to & \lambda_{2n-l-m} & & & & \\
\vdots & & \vdots & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\lambda_{l-m+1}+\lambda_1 & \to & \lambda_{l-m+2} & & & & & & \\
\downarrow & & & & & & & & \\
\lambda_{l-m} & & & & & & & &
\end{array}$$

where the representations are indicated by their highest weights, given in terms of the fundamental weights λ_i, and $\lambda_0\equiv 0$ indicates a singlet. (In addition, to keep the graph simple, the labels have been left out.) For $l+m>n$ the graph truncates at the $(n-l+1)$th column, since the representations to the right of this column in the graph are then no longer present in the decomposition of $l\otimes m$.

The S-matrix is then given by

$$S_{lm}(\theta)=\mathbf{P}s_{lm}(\theta)\sum_{p=0}^{\mathrm{Min}\,(n-l,m)}\ \sum_{q=0}^{m-p}\tau_{\lambda_{l+p-q}+\lambda_{m-p-q}}(\theta)P_{\lambda_{l+p-q}+\lambda_{m-p-q}} \tag{9}$$

where

$$\tau_{\lambda_{l+p-q}+\lambda_{m-p-q}}(\theta)=\prod_{p'=1}^{p}[2p'+l-m]\prod_{q'=1}^{q}[h+2q'-l-m] \tag{10}$$

with

$$[x]\equiv\frac{\theta+x\mathrm{i}\pi/h}{\theta-x\mathrm{i}\pi/h}\qquad\text{and}\qquad h=2n+2.$$

In (9), $s_{lm}(\theta)$ is an overall scalar factor which cannot be determined from the TPG. However, we can fix $s_{11}(\theta)$ to an overall CDD ambiguity by requiring that $S_{11}(\theta)$ be unitary and crossing-symmetric. A solution that achieves this in such a way that $S_{11}(\theta)$ has no poles in the physical strip $0 \leqslant \operatorname{Im} \theta \leqslant \pi$ is [4]

$$s_{11}(\theta) = k(\theta) = \frac{\Gamma\left(\frac{i\theta}{2\pi}\right)\Gamma\left(-\frac{i\theta}{2\pi}+\frac{1}{2}\right)\Gamma\left(-\frac{i\theta}{2\pi}+\frac{1}{h}\right)\Gamma\left(\frac{i\theta}{2\pi}+\frac{1}{h}+\frac{1}{2}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{i\theta}{2\pi}+\frac{1}{2}\right)\Gamma\left(\frac{i\theta}{2\pi}+\frac{1}{h}\right)\Gamma\left(-\frac{i\theta}{2\pi}+\frac{1}{h}+\frac{1}{2}\right)}.$$

Instead, however, we assume some bound state structure and choose a solution [4] which has a positive residue (direct channel) simple pole at $\theta = 2i\pi/h$ corresponding to the particle fusion $11 \to 2$, which is possible because at this value of θ the S-matrix projects only on to the 2-component of $1 \otimes 1$—as can easily be seen from the TPG for $1 \otimes 1$

$$\begin{tikzcd} 2\lambda_1 \arrow[r, "2i\pi/h"] \arrow[d, "i\pi"] & \lambda_2 \\ \lambda_0 & \end{tikzcd}$$

A solution which achieves this is then

$$s_{11}(\theta) = -(2)(h-2)k(\theta)$$

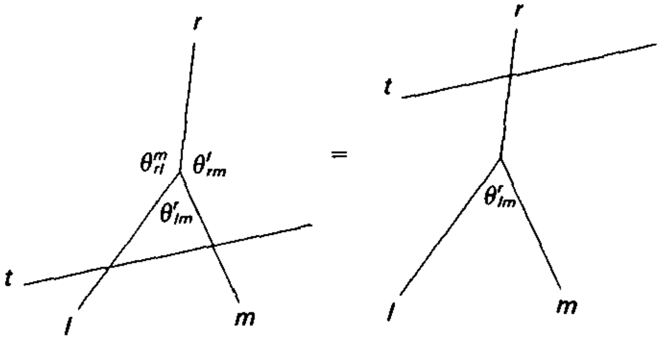
where we have used the notation [7]

$$(x) \equiv \frac{\sinh(\theta/2 + i\pi x/2h)}{\sinh(\theta/2 - i\pi x/2h)}.$$

In principle, we could now have deduced all the $S_{lm}(\theta)$ from $S_{11}(\theta)$ using the bootstrap principle, which implies that

$$S_{tr}(\theta) = (1 \otimes S_{tm}(\theta + i\bar{\theta}^{l}_{rm}))(S_{tl}(\theta - i\bar{\theta}^{m}_{rl}) \otimes 1)|_r$$

where $\bar{\theta} = \pi - \theta$ and $|_r$ indicates the restriction of the tensor product $l \otimes m$ to state r, or, schematically,



whenever $S_{lm}(\theta)$ has a positive residue simple pole at $\mathrm{i}\theta^r_{lm}$ with residue proportional to $r \subset l \otimes m$. However, in practice the calculations, which involve complex computations in Brauer's algebra (the centralizer algebra of c_n) are too complicated; we must instead use the TPG to give the matrix structure. However, we *can* use the bootstrap to determine the scalar factors $s_{lm}(\theta)$. We then have

$$s_{lm}(\theta) = X_{lm}(\theta) k_{lm}(\theta) \tag{11}$$

where

$$k_{lm}(\theta) = \prod_{\substack{p=1-m \\ \text{step } 2}}^{m-1} \prod_{\substack{q=1-n \\ \text{step } 2}}^{n-1} k\left(\theta + \frac{p+q}{h} \mathrm{i}\pi\right) \tag{12}$$

and the X_{lm}, which give all the pole structure of the S-matrices, have in fact already been calculated in the context of purely elastic scattering theories (in which the particles do not form multiplets), where they are related to $d^{(2)}_{n+1}$ affine Toda theories [7]:

$$X_{lm}(\theta) = \prod_{\substack{p=l-m+1 \\ \text{step } 2}}^{l+m-1} (p-1)(p+1)(h-p-1)(h-p+1). \tag{13}$$

Thus the complete set of $S_{lm}(\theta)$ is given by substituting (10) and (11)–(13) into (9).

Since the matrix structure of S_{lm} has been found by a method other than the bootstrap, it is worth examining how the bootstrap would work for our solutions. As in the S_{11} case, we can look at the TPG for $l \otimes m$ and, seeing that the representations $l-m$ and (for $l+m \leqslant n$) $l+m$ are connected to the rest of the graph by a single edge whose label is valued in the physical strip, we expect S_{lm} to have bootstrap poles corresponding to these fusings. This is indeed what we find: such fusings correspond precisely to the positive residue simple poles in X_{lm}, and the bootstrap procedure on these poles thus closes on the expected spectrum of n massive multiplets. This correspondence is really quite remarkable—facts about solutions of the YBE are being predicted by the bootstrap structure of *scalar* functions X_{lm}. We can think of the fusings as being due to a three-point coupling between the particles l, m and $l+m$ with

$$\theta^{l+m}_{lm} = \frac{l+m}{h}\pi \qquad \theta^{l}_{l+m\,m} = \frac{h-l}{h}\pi \qquad \theta^{m}_{l+m\,l} = \frac{h-m}{h}\pi \tag{14}$$

and then the fact that $\theta^{l+m}_{lm} + \theta^{l}_{l+m\,m} + \theta^{m}_{l+m\,l} = 2\pi$ is a highly non-trivial consequence of Yangian representation theory, proved only in the very special case of a three-point coupling between identical particles [8].

There is, however, a subtlety in that there are also positive residue cubic poles in $X_{lm}(\theta)$ for which the residue of $S_{lm}(\theta)$ does not correspond to a fundamental representation or indeed to any subgraph of the TPG. We therefore expect that such poles should not be interpreted as bootstrap poles, yet we know from affine Toda theories [7] and the d_4 Yangian-invariant theory [9] that such poles *can* mask simple poles in particle states and thus form part of the bootstrap. Thus, at present, we are forced to fall back on the *postulate* that the spectrum of the theory consists only of particles in fundamental representations of the Yangian. It would therefore be nice to have an independent, quantum field theoretic way of deducing whether or not a given non-simple pole in

$S_{lm}(\theta)$ should be included in the bootstrap. Whereas in affine Toda theories direct comparison with perturbation theory is possible [7], for models with multiplet structure (such as the principal chiral model) matters are more complicated, and methods such as the $(1/N)$-expansion do not seem promising.

The spectrum of masses in the theory can be deduced from conservation of momentum at bootstrap poles, and for the c_n theories has been deduced [4] from $S_{1m}(\theta)$:

$$m_p = m \sin\left(\frac{p\pi}{h}\right) \qquad p = 1, \ldots, n. \tag{15}$$

An interesting alternative way of deducing the values of the bootstrap poles and thus the mass spectrum has been proposed recently by Belavin [10] and used by him to compute the a_n mass spectrum. He considered the two commuting conserved charges $Q_0^a Q_0^a (= C_2)$ and $Q_1^a Q_0^a$ and applied the bootstrap principle: if the residue of a pole of $S_{lm}(\theta)$ is a particle state r then

$$\Delta(Q_0^a)\Delta(Q_0^a) l \otimes m = Q_0^a Q_0^a l \otimes m|_r$$

$$\Delta(Q_1^a)\Delta(Q_0^a) l \otimes m = Q_1^a Q_0^a l \otimes m|_r.$$

Using the coproducts (1), (2) and the representation (7), and setting $\theta_r = 0$, it is then possible to deduce the value of $\mathrm{i}\theta_{lm}^r = \theta_l - \theta_m$:

$$\theta_{lm}^r = \frac{2\tilde{r}(\tilde{l} + \tilde{m} - \tilde{r})}{2(\tilde{l}\tilde{r} + \tilde{m}\tilde{r} + \tilde{l}\tilde{m}) - (\tilde{l}^2 + \tilde{m}^2 + \tilde{r}^2)} \pi$$

where we have written $\tilde{p} \equiv C_2(p)$. When we apply this method to the c_n case we obtain the expected fusing angles (14) and thus (15); Belavin's method also works in this way for all particle multiplets (for any $\mathscr{A}$) which are irreducible as representations of $\mathscr{A}$.

If we wish to develop the methods discussed in this letter as an alternative to the bootstrap procedure for calculating factorized S-matrices for general $\mathscr{A}$, it is clear that new results on representations of Yangians are needed†: both the TPG and Belavin's method depend crucially on the explicit action (7) of Q_0^a and Q_1^a on the particles. Apart from $v = V$, the only case for which such an action is known is Drinfeld's construction [2] of $v = \text{adjoint} \oplus \text{singlet}$; the corresponding R-matrices have been constructed by Chari and Pressley [8], although it is not clear how to extend Belavin's method to this representation.

Finally, it seems that neither the bootstrap (which describes the decomposition of tensor products of $Y(\mathscr{A})$ representations) nor the methods [5, 8] for solving (6) give any general insight into the mass spectra and fusings obtained. Since, as we mentioned above, much information about the YBE is already contained in the X_{lm}, and since the mass spectra and fusings given by the X_{lm} have a beautiful description in terms of root systems of Lie algebras [12] (at least for simply-laced $\mathscr{A}$; for non-simply-laced $\mathscr{A}$ the situation is more complicated), it therefore appears that there is every prospect of rich undiscovered structure in the representation theory of the Yangian.

NJM acknowledges financial support under a UK SERC Studentship.

† At this point we should note that recent results [11] on off-shell, infinite-dimensional representations of dynamical Yangian symmetry do not seem to help with this.

References

[1] Lüscher M 1978 *Nucl. Phys.* B **135** 1
de Vega H J, Eichenherr H and Maillet J M 1984 *Nucl. Phys.* B **240** 377
Bernard D 1991 *Commun Math Phys.* **137** 191
[2] Drinfeld V G 1985 *Sov. Math. Dokl.* **32** 254
[3] Kulish P P, Sklyanin E K and Reshetikhin N Y 1981 *Lett. Math. Phys.* **5** 393
[4] Ogievetsky E, Reshetikhin N Y and Wiegmann P 1986 *Nucl. Phys.* B **280** 45
[5] MacKay N J 1991 *J. Phys. A: Math. Gen.* **24** 4017
Zhang, R B , Gould M D and Bracken A J 1991 *Nucl. Phys.* B **354** 625
[6] Berg B, Karowski M, Weisz and Kurak V 1978 *Nucl. Phys.* B **134** 125
[7] Braden H W, Corrigan E, Dorey P E and Sasaki R 1989 *Phys. Lett.* **227B** 441; 1990 *Nucl. Phys.* B **338** 689
Christe P and Mussardo G 1989 *Nucl. Phys.* B **330** 465; 1990 *Int. J. Mod. Phys.* A **5** 4581
[8] Chari V and Pressley A 1991 *J. Reine Angew. Math.* **417** 87
[9] MacKay N J 1991 *Nucl. Phys.* B **356** 729;
1992 *PhD Thesis*, unpublished
[10] Belavin A A 1992 *Phys. Lett.* **283B** 67
[11] LeClair A and Smirnov F A 1992 *Int. J. Mod. Phys.* A **7** 2997
[12] Dorey P E 1991 *Nucl. Phys.* B **358** 654